ON ORDER PRESERVING CONTRACTIONS*

BY
S. R. FOGUEL

ABSTRACT

Let (Ω,Σ,μ) be a measure space and let P be an operator on $L_2(\Omega,\Sigma,\mu)$ with $\|P\| \leqq 1$, $Pf \geqq 0$ a.e. whenever $f \geqq 0$. If the subspace K is defined by

$$K = \{x \mid \|P^n x\| = \|P^{*n} x\| = \|x\|, \quad n = 1,2,\cdots\}$$

then $K = L_2(\Omega,\Sigma_1,\mu)$, where $\Sigma_1 \subset \Sigma$ and on K the operator P is "essentially" a measure preserving transformation. Thus the eigenvalues of P of modulus one, form a group under multiplication.

This last result was proved by Rota for finite μ here finiteness is not assumed) and is a generalization of a theorem of Frobenius and Perron on positive matrices.

Introduction. The purpose of this note is to generalize the results of [2]. In [2] Rota studies the eigenvalues of modulus one of a contraction P on $L_1(S, \Sigma, \mu)$ where μ is a finite measure and P satisfies the following:

a. $Pf \geqq 0$ whenever $f \geqq 0$.

b. ess. sup. $|Pf| \leqq$ ess. sup. $|f|$.

This problem is related to the Frobenius Perron Theory. For bibliography on the subject we refer to [2].

Our generalization is two-fold:

1. The measure μ is not assumed to be finite.

2. The operator P is a contraction on $L_2(S,\Sigma,\mu)$ and is not assumed to be defined on $L_1(S,\Sigma,\mu)$.

If P has norm one in $_1$ and L_∞ then, by the Riesz Convexity Theorem it has norm 1 also over L_2, thus 2 is weaker than Rota's assumption.

We shall use the method of the proof of Theorem 2.2 and Lemma 1.2 of [1]. There the case $\mu(S) < \infty$ was studied.

The results of [2] are included in Theorems 1 and 3 of this note.

Let (S,Σ,μ) be a measure space with $\mu \geqq 0$.

LEMMA 1. *Let L be a closed subspace of $L_2(S,\Sigma,\mu)$, which satisfies:*

Received March 28, 1963.

* The research reported in this document has been sponsored in part by Air Force Office of Scientific Research, OAR through the European Office, Aerospace Research, United States Air Force.

(1) *If $f \in L$ then $Re\, f \in L$.*

(2) *If f is real and $f \in L$ then $|f| \in L$.*

(3) *If $f \geqq 0$ a.e., $f \in L$, and c is a positive constant then* $\min(f,c) \in L$.

Let Σ' contain all the sets, in Σ, whose characteristic functions are in L; then

(a) *The sets in Σ' form a field;*

(b) *The characteristic functions of sets in Σ' span L.*

Proof. Let f, g be real valued functions in L. Then

$$\max(f,g) = \tfrac{1}{2}(|f-g| + f + g) \in L$$

$$\min(f,g) = \tfrac{1}{2}(f + g - |f-g|) \in L.$$

If σ and τ are in Σ', let $I(\sigma)$ and $I(\tau)$ denote their characteristic functions. Then $\max(I(\sigma), I(\tau)) \in L$ and $\min(I(\sigma), I(\tau)) \in L$ or: *$\sigma \cup \tau \in \Sigma'$ and $\sigma \cap \tau \in \Sigma'$.* In order to prove (b) it is enough to show that the only function in L orthogonal to Σ' is the zero function. Now if $f \in L$ is orthogonal to all functions $I(\sigma), \sigma \in \Sigma'$, then so is $Re\, f$. Thus we may assume that f is real. Let $f_+ = \tfrac{1}{2}(|f| + f) \in L$ and let c be a positive constant. Then, by (3), $\min(f_+, c) \in L$ and also $f_+ - \min(f_+, c)$ $g \in L$.

Let $\phi = c^{-1}\min(f_+, c)$. Then $h_\varepsilon = \varepsilon^{-1}\min(\varepsilon\phi, g) \in L$. *Now*:

$$h_\varepsilon(\omega) = 0 \text{ if } f_+(\omega) \leqq c, \text{ since then } g(\omega) = 0,$$

while:
$h_\varepsilon(\omega) = 1$ if $f_+ \geqq c + \varepsilon$. Also, for every ω, $0 \leqq h_\varepsilon(\omega) \leqq 1$. Hence $h_\varepsilon(\omega)$ tends to the characteristic function of $\{\omega | f_+(\omega) > c\}$ as $\varepsilon \to 0$, thus $I\{\omega | f_+(\omega) > c\} \in L$ and is orthogonal to f; i.e., $f \leqq c$ a.e. for every $c > 0$. Therefore $f_+ = 0$ a.e. Applying the same argument to $-f$ we get $f = 0$.

Remark. If $\mu(\Omega) < \infty$ then $1 \in L_2$ and Condition (3) is a consequence of Condition 2 and

(3′) $$1 \in L.$$

Definition 1. *An operator P on $L_2(S, \Sigma, \mu)$ is called an order preserving contraction (O.P.C.) provided:*

1. If $f \in L_2$ is real valued then so is Pf.
2. If $0 \leqq f \in L_2$ a.e. then $Pf \geqq 0$ a.e.
3. If $f \in L_2$ is real valued and $f \leqq c$ a.e. then $Pf \leqq c$ and $P^*f \leqq c$.
4. $\|P\| \leqq 1$.

Note that the conditions (1) (2) and (4) are the same if we replace P *by* P^*. Throughout this note P will always be an *O.P.C.*

LEMMA 2. *If* $Pf = e^{i\theta}f$ *then* $P|f| = |f|$.

Proof. We will need the inequality

$$|PF| \leqq P|F| \text{ a.e.}$$

Now if F is real this is immediate. Generally we have

$$|Re\,PF| \leqq P|F| \text{ a.e. since } \pm Re\,F \leqq |F|.$$

Also for every λ with $|\lambda| = 1$

$$(*) \qquad |Re\lambda PF| \leqq P|\lambda F| = P|F| \text{ a.e.}$$

Thus if $|PF| > P|F|$ on a set of positive measure there is a set σ of positive measure, such that if $\omega \in \sigma$ then

$$|PF| > (1+\delta)P|F|, \qquad |\arg PF - \phi| < \varepsilon$$

where $\delta > 0$ $\varepsilon > 0$ and ε can be chosen arbitrarily small. But then

$$Re|e^{-i\phi}PF| > |PF| \cos\varepsilon > |F| \qquad \text{contradicting } (*).$$

(This argument was suggested to us by Y. Katznelson). The proof of the Lemma is now straightforward:

$$\|f\|^2 \geqq \|P|f|\| \cdot \|f\| \geqq (P|f|,|f|) \geqq |(Pf,f)| = \|f\|^2$$

hence $(P|f|,|f|) = \|P|f|\| \, \|f\|$ or $P|f| = |f|$.

THEOREM 1. *Let* $L = \{f \mid Pf = f\}$ *and let* Σ' *contain all the sets* σ *in* Σ *such that* $I(\sigma) \in L$. *Then* Σ' *is a field and its characteristic functions generate* L.

Proof. It is enough to verify Conditions (1), (2) and (3) of Lemma 1. The first condition is obviously satisfied. Now if $|f| \in L$ then $|f| \in L$ by Lemma 2. Finally if $0 \leqq f = Pf$ then

$$P[\min(f,c)] \leqq Pf = f, \quad P[\min(f,c)] \leqq c;$$

thus

$$P[\min(f,c)] \leqq \min(f,c).$$

Hence

$$P[f - \min(f,c)] = f - P[\min(f,c)] \geqq f - \min(f,c)$$

and $f - \min(f,c) \geqq 0$. We must have equality a.e., since $\|P\| \leqq 1$, thus $\min(f,c) \in L$.

DEFINITION 2. $K = \{f \mid \|P^n f\| = \|P^{*n} f\| = \|f\|, \; n \geqq 1\}$.

Now $\| P^n f \| = \|f\|$ if and only if $P^{*n}P^n f = f$, and it is easy to check that $P^{*n}P^n$ is an $O.P.C.$ Also $\| P^{*n} f \| = \|f\|$ if and only if $P^n P^{*n} f = f$. Thus K is generated by characteristic functions of the intersections of the corresponding subfields of Σ.

DEFINITION 3. *Let Σ_1 contain all the sets, σ, such that $I(\sigma) \in K$.*

By the above remarks Σ_1 is a field and it generates K.

THEOREM 2. *The set K is a closed subspace of L_2, invariant under P and P^*. On K, P is a unitary operator. If $f \perp K$ then*

$$\text{weak lim } P^n f = \text{weak lim } P^* f = 0.$$

Also, if $\sigma \in \Sigma_1$, then $PI(\sigma) = I(\tau)$, where $\tau \in \Sigma_1$.

Proof. It is enough to prove the last statement since the rest is proved in Theorem 1.1 of [1]. Let $\sigma \in \Sigma_1$ and $PI(\sigma) = f$ then $0 \leqq f \leqq 1$. Let $\tau_1 \in \Sigma$ be such that if $\omega \in \tau_1$ then $0 < f(\omega) < 1$, and let $g = (1 - f).I(\tau_1)$. Then $0 \leqq g \leqq 1$, and $g + f \leqq 1$. Thus $P^*(f + g) \leqq 1$ but $P^* f = I(\sigma)$; hence $P^* g(\omega) = 0$ if $\omega \in \sigma$ or $0 = (I(\sigma), P^* g) = (P^* f, P^* g) = (f, g)$. Therefore, τ_1 is a set of measure zero or f is a characteristic function (necessarily of a set in Σ_1).

LEMMA 3. *If f,g are in K and are real valued, then $P[\min(f,g)] = \min(Pf, Pg)$. If, in addition, f is bounded then $P(f.g) = Pf Pg$.*

Proof. Since P is order preserving

$$P[(\min(f,g)] \leqq \min(Pf, Pg),$$

and a similar relation holds for P^*. Thus $P^*[\min(Pf, Pg)] \leqq \min(f,g)$. Applying P to this inequality we get $P[\min(f,g)] \leqq \min(Pf, Pg) \leqq P[\min(f,g)]$. In particular if f and g are characteristic functions then

$$P(fg) = Pf \cdot Pg.$$

The last part of the Lemma is proved by taking limits of sums of characteristic functions.

THEOREM 3. *If $Pf = e^{i\theta} f$ then*

$$P[(\text{sgn} f)^2 |f|] = e^{2i\theta}[(\text{sgn} f)^2 |f|].$$

Proof. It is well known that if $Pf = e^{i\theta} f$ and $\| P \| \leqq 1$ then

$$P^* f = e^{-i\theta} \quad ((P^* f, f) = e^{-i\theta} \|f\|^2).$$

Thus $f \in K$ and, by Lemma 2, $P|f| = |f|$. Let $I_\varepsilon = I\{\omega \,|\, |f(\omega)| \geqq \varepsilon\}$. Then $PI_\varepsilon = I_\varepsilon$ by Theorem 1; thus

$$P[(\text{sgn} f) I_\varepsilon |f|] = PI_\varepsilon Pf = I_\varepsilon e^{i\theta} f = e^{i\theta} I_\varepsilon (\text{sgn} f) |f|.$$

On the other hand

$$P[(\text{sgn}f)I_\varepsilon|f|] = I_\varepsilon P[(\text{sgn}f)I_\varepsilon]P|f| = I_\varepsilon|f|P(I_\varepsilon \text{sgn}f)$$

or

$$P(I_\varepsilon \text{sgn}f) = I_\varepsilon P(I_\varepsilon \text{sgn}f) = e^{i\theta}I_\varepsilon \text{sgn}f.$$

Therefore

$$P[(\text{sgn}f)^2\, I_\varepsilon\, |f|] = |f|\, P[(\text{sgn}f)I_\varepsilon]^2 = f e^{2i\theta}I_\varepsilon(\text{sgn}f)^2.$$

Let $\varepsilon \to 0$, then $I_\varepsilon|f| \to |f|$; hence

$$P[(\text{sgn}f)^2|f|] = e^{2i\theta}(\text{sgn}f)^2|f|.$$

Let us conclude with a uniqueness theorem.

THEOREM 4. *Let P_1 and P_2 be unitary order preserving operators. Then the subspace $L = \{f \mid P_1f = P_2f\}$ is generated by the characteristic functions contained in it.*

Proof. We will verify the three conditions of Lemma 1:

(1) If $P_1f = P_2f$ then $P_1(Re\, f) = Re\, P_1f = Re\; P_2f = P_2(Re\, f)$.
(2) If f *is real valued and* $P_1f = P_2f$ *then*

$$P_1(f_+) = (P_1f)_+ = (P_2f)_+ = P_2(f_+)$$

and

$$P_1(f_-) = (P_1f)_- = (P_2f)_- = P_2(f_-).$$

Since P_if_+ and P_if_- are positive, their sum is P_if and $(P_if_+, P_if_-) = (f_+, f_-) = 0$.
(3) It will be enough to show that

$$P_i[\min(f,c)] = \min(P_if,c)$$

for $f \geqq 0$ *and* c a positive constant. Now

$$P_i[\min(f,c)] \leqq \min(P_if,c);$$

thus for P_i^* we have

$$P_i^*[\min(P_if,c)] \leqq \min(f,c).$$

Applying P_i we get

$$\min(P_if,c) \leqq [P_i \min(f,c)].$$

Thus, in order to find whether two unitary order preserving operators are equal, it is enough to show that $P_1f = P_2f$, whenever f is a function such that the functions $I\{\omega \mid f(\omega) \in A\}$ generate $L_2(S,\Sigma,\mu)$.

BIBLIOGRAPHY

1. Foguel, S.R., Powers of a contraction in Hilbert space, To be published in the *Pacific J. of Math.*

2. Rota, G.C., 1961, On the eigenvalues of positive operators, *Bull. Amer. Math. Soc.*, **67**, 556–558.

THE HEBREW UNIVERSITY OF JERUSALEM